

ANALYTIC GEOMETRY

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PREFACE

In adding another text-book on analytic geometry to the large list of such books now available, the authors feel that a statement of the reasons for this one is desirable. In compiling the work they have endeavored to keep constantly in view the needs of that large class of students in our colleges and technical schools who, though in a way sympathetic toward mathematical study, are yet lacking in that quick appreciation of the mathematical point of view which is a characteristic of the so-called mathematical mind. The assimilation of new ideas, or even of new combinations and applications of ideas already familiar, in which the study of analytic geometry especially abounds, is not easy even for the trained mind, and for the student approaching the subject for the first time is the source of most of his difficulties. The attempt here is to make this approach by easy gradations so that the student may not lose himself on the way, but may feel sure of his ground as he advances. To try to gain this end by the lack of rigorous demonstration would be to lose sight of an important object of such study, which, rightly viewed, is to assist the student to an acquaintance with correct principles of mathematical reasoning and the accompanying scientific attitude of mind. The aim has been therefore to preserve mathematical rigor throughout.

As to details, much of the book necessarily follows well established lines of procedure. The experienced teacher will find, however, here and there, some departure from these lines which it is hoped will be approved. Owing to the increasing use of the imaginary, and its growing importance to the student of both pure and applied mathematics, some elementary dis-

cussion of imaginary elements in geometry has been included, which the authors believe will be of value in accustoming the student to look upon the complex number as a useful member of the number system. The lists of exercises have been prepared with especial care. Throughout the text short lists have been inserted where needed for the immediate illustration of principles or methods. At the conclusion of each chapter longer lists are inserted. These are divided into two parts, "normal exercises," which are direct applications of the text of the chapter, and "general exercises," which are designed to give the student opportunity of testing his grasp of the subject as a whole, and of its underlying principles. There will be found in each set of "normal exercises" one or two examples illustrating each separate point of the theory developed in the corresponding portion of the text, so far as it can be illustrated by exercises. The student may therefore be assured that he has made application of all of the theory when he shall have worked *all* the normal exercises.

While the book is to be regarded as a text-book of plane analytic geometry, the concluding chapter is devoted to solid geometry. Only the barest outline of some of the fundamental principles of this subject is included, enough to enable the student, when he studies calculus, and wishes to apply its principles to problems involving solids and surfaces, to feel that he is not on entirely new ground.

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TABLE OF CONTENTS

	PAGE
FORMULAS, TABLES, ETC., FOR REFERENCE.....	vii
CHAPTER I. CARTESIAN COORDINATES	1
“ II. GRAPHICAL REPRESENTATION OF EQUA- TIONS	15
“ III. THE STRAIGHT LINE.....	40
“ IV. THE CIRCLE.....	65
“ V. THE CONIC SECTIONS.....	75
“ VI. TANGENTS, NORMALS, DIAMETERS, POLES AND POLARS.....	107
“ VII. APPLICATIONS OF ANALYTIC GEOMETRY....	127
“ VIII. THE GENERAL EQUATION OF THE SECOND DEGREE.....	142
“ IX. HIGHER PLANE CURVES, PARAMETRIC EQUATIONS.....	161
“ X. POLAR COORDINATES.....	179
“ XI. EMPIRICAL EQUATIONS.....	191
“ XII. EXTENSION OF COORDINATE GEOMETRY TO SOME SPACE PROBLEMS.....	205
APPENDIX. CURVES FOR REFERENCE ..	236

FORMULAS, TABLES, ETC., FOR REFERENCE

A few important definitions, formulas, and tables, from elementary algebra and trigonometry, which will be needed in the study of this book, are placed here for convenience of reference.

A. Algebraic definitions and formulas

I. DEFINITIONS

(a) The **degree of a term** in specified letters is the number of times these letters occur as factors in that term. Thus ab^2c^3d is of the seventh degree in a , b , c , and d ; $a^2x^2y^2$ is of the fourth degree in x and y .

(b) The **degree of a polynomial** in specified letters is defined as that of the term of highest degree in those letters. Thus $ax^3 + bx^2y + cx^2 + dy + e$ is of the third degree in x and y .

(c) A polynomial is said to be **homogeneous** in specified letters when each of its terms is of the same degree in those letters. Thus $ax^2 + 5xy - b^2y^2$ is homogeneous and of the second degree in x and y .

(d) In an equation containing one or more variables the term which contains no variable factors is called the **absolute term**. Thus in $ax^2 + by^2 - a^2c^2 = 0$, where x and y are the variables, the absolute term is $-a^2c^2$. In $x + y = 0$, the absolute term is zero.

(e) A **root of an equation** is a value of the variable which, when substituted for the variable, satisfies the equation.

II. THE QUADRATIC EQUATION

(a) The roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad (1)$$

are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (2)$$

(b) The roots of the quadratic (1) are

$$\left. \begin{array}{ll} \text{real and unequal if } b^2 > 4ac, \\ \text{real and equal} & \text{if } b^2 = 4ac, \\ \text{imaginary} & \text{if } b^2 < 4ac. \end{array} \right\} \quad (3)$$

(c) The sum of the roots of the quadratic (1) is

$$r_1 + r_2 = -\frac{b}{a}. \quad (4)$$

(d) The product of the roots of the quadratic (1) is

$$r_1 r_2 = \frac{c}{a}. \quad (5)$$

(e) If the coefficient a in the quadratic (1) varies and approaches zero, the coefficients b and c remaining constant, one root of the equation increases without limit.

III. FACTORS OF $Ax^2 + Hxy + By^2$

A homogeneous expression of the second degree in x and y ,

$$Ax^2 + Hxy + By^2, \quad (6)$$

can always be factored. The factors are conveniently expressed thus

$$\frac{1}{4A} [2Ax + (H + \sqrt{H^2 - 4AB})y][2Ax + (H - \sqrt{H^2 - 4AB})y]. \quad (7)$$

IV. IMAGINARIES

(a) $\sqrt{-1}$ is called the imaginary unit, and is usually represented by the letter i .^{*} Thus $x + iy$ means $x + y\sqrt{-1}$.

(b) An expression containing both real and imaginary terms is called a complex expression.

(c) If two complex expressions are equal, the real parts of the two expressions are equal, and the imaginary parts are equal. Thus

$$\left. \begin{array}{l} \text{if } x + iy = a + ib, \text{ then } x = a, \text{ and } y = b, \\ \text{if } x + iy = 0, \text{ then } x = 0, \text{ and } y = 0. \end{array} \right\} \quad (8)$$

V. LOGARITHMS

(a) If $y = a^x$, x is called the logarithm of y to the base a . This equation may therefore be written in the form $x = \log_a y$.

(b) In the case of the so-called common logarithms, the logarithms used in computation, the base is 10.

^{*} In electrical theory j is used instead of i to represent $\sqrt{-1}$.

(c) Logarithms used algebraically are usually Napierian, or natural logarithms. The base of this system of logarithms is the incommensurable number $e = 2.7182818285 \dots$, called the Napierian base, and defined by the infinite series

$$e = 1 + 1 + \frac{1}{\underline{2}} + \frac{1}{\underline{3}} + \frac{1}{\underline{4}} + \frac{1}{\underline{5}} + \dots \quad (9)$$

(d) When in algebraic work a logarithm is written without any expressed base, e. g., $\log (1 + x^2)$, the base is understood to be e .

B. Trigonometric formulas

If l is the length, and r the radius of the arc subtending the angle of θ radians,

$$l = r\theta. \quad (10)$$

$$180^\circ = \pi \text{ radians}. \quad (11)$$

$$\csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta}, \quad \sec \theta = \frac{1}{\cos \theta}. \quad (12)$$

$$\sin^2 \theta + \cos^2 \theta = 1. \quad (13)$$

$$\sec^2 \theta = 1 + \tan^2 \theta. \quad (14)$$

$$\csc^2 \theta = 1 + \cot^2 \theta. \quad (15)$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}. \quad (16)$$

If $\tan \theta = \frac{m}{n}$, then

$$\sin \theta = \pm \frac{m}{\sqrt{m^2 + n^2}}, \quad \cos \theta = \pm \frac{n}{\sqrt{m^2 + n^2}}. \quad (17)$$

$$\cos \theta = \sin (90^\circ \pm \theta), \quad \cot \theta = \begin{cases} \tan (90^\circ - \theta), \\ -\tan (90^\circ + \theta). \end{cases} \quad (18)$$

$$\begin{aligned} \sin (180^\circ - \theta) &= \sin \theta, & \cos (180^\circ - \theta) &= -\cos \theta, \\ \tan (180^\circ - \theta) &= -\tan \theta. \end{aligned} \quad (19)$$

$$\begin{aligned} \sin (180^\circ + \theta) &= -\sin \theta, & \cos (180^\circ + \theta) &= -\cos \theta, \\ \tan (180^\circ + \theta) &= \tan \theta. \end{aligned} \quad (20)$$

$$\sin (-\theta) = -\sin \theta, \quad \cos (-\theta) = \cos \theta, \quad \tan (-\theta) = -\tan \theta. \quad (21)$$

$$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y. \quad (22)$$

$$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y. \quad (23)$$

$$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}. \quad (24)$$

$$\sin 2\theta = 2 \sin \theta \cos \theta. \quad (25)$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta. \quad (26)$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \quad (27)$$

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta, \quad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta. \quad (28)$$

$$1 - \cos \theta = 2 \sin^2 \frac{1}{2}\theta, \quad 1 + \cos \theta = 2 \cos^2 \frac{1}{2}\theta. \quad (29)$$

$$\tan \frac{1}{2}\theta = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}. \quad (30)$$

In any plane triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \quad (31)$$

$$a^2 = b^2 + c^2 - 2bc \cos A, \text{ etc.} \quad (32)$$

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

	First Quadrant	Second Quadrant	Third Quadrant	Fourth Quadrant
$\sin \theta$ $\csc \theta$	+	+	-	-
$\cos \theta$ $\sec \theta$	+	-	-	+
$\tan \theta$ $\cot \theta$	+	-	+	-

C. Values of the trigonometric functions of angles

I. *Certain exact values of these functions*

Angle in Degrees	Angle in Radians	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	∞	1	∞
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	$\frac{1}{2}\pi$	1	0	∞	0	∞	1

II. *Decimal values of the trigonometric functions for each 5° of the quadrant*

Angle in Degrees	Angle in Radians	sin	cos	tan	cot	sec	csc
0°	0	0.000	1.000	0.000	∞	1.000	∞
5°	0.087	0.087	0.996	0.087	11.430	1.004	11.474
10°	0.175	0.174	0.985	0.176	5.671	1.015	5.759
15°	0.262	0.259	0.966	0.268	3.732	1.035	3.864
20°	0.349	0.342	0.940	0.364	2.747	1.064	2.924
25°	0.436	0.423	0.906	0.466	2.145	1.103	2.366
30°	0.524	0.500	0.866	0.577	1.732	1.155	2.000
35°	0.611	0.574	0.819	0.700	1.428	1.221	1.743
40°	0.698	0.643	0.766	0.839	1.192	1.305	1.556
45°	0.785	0.707	0.707	1.000	1.000	1.414	1.414
50°	0.873	0.766	0.643	1.192	0.839	1.556	1.305
55°	0.960	0.819	0.574	1.428	0.700	1.743	1.221
60°	1.047	0.866	0.500	1.732	0.577	2.000	1.155
65°	1.134	0.906	0.423	2.145	0.466	2.366	1.103
70°	1.222	0.940	0.342	2.747	0.364	2.924	1.064
75°	1.309	0.966	0.259	3.732	0.268	3.864	1.035
80°	1.396	0.985	0.174	5.671	0.176	5.759	1.015
85°	1.484	0.996	0.087	11.430	0.087	11.474	1.004
90°	1.571	1.000	0.000	∞	0.000	∞	1.000

D. Miscellaneous tables*I. Degrees and Radians*

$$1^\circ = 0.0174533 \text{ rad.}, \quad 1' = 0.0002909 \text{ rad.}, \quad 1'' = 0.0000048 \text{ rad.}$$

$$1 \text{ rad.} = 57^\circ.29578 = 57^\circ 17' 44''.8 = 3437'.75 = 206264''.8$$

$$\begin{array}{l} 0.1 \text{ rad.} = 5^\circ 43' 46''.5 \quad 0.4 \text{ rad.} = 22^\circ 55' 5''.9 \quad 0.7 \text{ rad.} = 40^\circ 6' 25''.4 \\ 0.2 \text{ " } = 11^\circ 27' 33''.0 \quad 0.5 \text{ " } = 28^\circ 38' 52''.4 \quad 0.8 \text{ " } = 45^\circ 50' 11''.8 \\ 0.3 \text{ " } = 17^\circ 11' 19''.4 \quad 0.6 \text{ " } = 34^\circ 22' 38''.9 \quad 0.9 \text{ " } = 51^\circ 33' 58''.3 \end{array}$$

II. Mantissas of the common logarithms of numbers

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	519	531	544	556	568	580	591
4	602	613	623	633	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	929	934	940	944	949
9	954	959	964	968	973	978	982	987	991	996

III. Napierian logarithms of numbers

N	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	$-\infty$	-2.30	-1.61	-1.20	-0.92	-0.69	-0.51	-0.36	-0.22	-0.11
1	0.00	0.10	0.18	0.26	0.34	0.41	0.47	0.53	0.59	0.64
2	0.69	0.74	0.79	0.83	0.88	0.92	0.96	0.99	1.03	1.06
3	1.10	1.13	1.16	1.19	1.22	1.25	1.28	1.31	1.34	1.36
4	1.39	1.41	1.44	1.46	1.48	1.50	1.53	1.55	1.57	1.59
5	1.61	1.63	1.65	1.67	1.69	1.70	1.72	1.74	1.76	1.77
6	1.79	1.81	1.82	1.84	1.86	1.87	1.89	1.90	1.92	1.93
7	1.95	1.96	1.97	1.99	2.00	2.01	2.03	2.04	2.05	2.07
8	2.08	2.09	2.10	2.12	2.13	2.14	2.15	2.16	2.17	2.19
9	2.20	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29
10	2.30	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39

IV. *Positive powers of e (Napierian anti-logarithms)*

Exp.	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.00	1.11	1.22	1.35	1.49	1.65	1.82	2.01	2.23	2.46
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	6.05	6.69
2	7.39	8.17	9.03	9.97	11.0	12.2	13.5	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	81.5	90.0	99.5	110.	122.	134.
5	148.	164.	181.	200.	221.	245.	270.	299.	330.	365.
6	403.	446.	493.	545.	602.	665.	735.	812.	898.	992.

V. *Negative powers of e (Napierian anti-logarithms)*

Exp.	0	-.1	-.2	-.3	-.4	-.5	-.6	-.7	-.8	-.9
-5	0.01	0.01	0.01	0.00						
-4	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
-3	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02
-2	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.07	0.06	0.06
-1	0.37	0.33	0.30	0.27	0.25	0.22	0.20	0.18	0.17	0.15
0	1.00	0.90	0.82	0.74	0.67	0.61	0.55	0.50	0.45	0.41

VI. *Square roots of numbers*

N	0	1	2	3	4	5	6	7	8	9
0	000	1.00	1.41	1.73	2.00	2.24	2.45	2.65	2.83	3.00
1	3.16	3.32	3.46	3.61	3.74	3.87	4.00	4.12	4.24	4.36
2	4.47	4.58	4.69	4.80	4.90	5.00	5.10	5.20	5.29	5.39
3	5.48	5.57	5.66	5.74	5.83	5.92	6.00	6.08	6.16	6.24
4	6.32	6.40	6.48	6.56	6.63	6.71	6.78	6.86	6.93	7.00
5	7.07	7.14	7.21	7.28	7.35	7.42	7.48	7.55	7.62	7.68
6	7.75	7.81	7.87	7.94	8.00	8.06	8.12	8.19	8.25	8.31
7	8.37	8.43	8.49	8.54	8.60	8.66	8.72	8.77	8.83	8.89
8	8.94	9.00	9.06	9.11	9.17	9.22	9.27	9.33	9.38	9.43
9	9.49	9.54	9.59	9.64	9.70	9.75	9.80	9.85	9.90	9.95

he Greek alphabet

A, α ,	I, ι , iota	P, ρ , rho
B, β ,	K, κ , kappa	Σ , σ , s , sigma
Γ , γ ,	Λ , λ , lambda	T, τ , tau
Δ , δ ,	M, μ , mu	Υ , υ , upsilon
E, ϵ ,	N, ν , nu	Φ , ϕ , phi
Z, ζ ,	Ξ , ξ , xi	X, χ , chi
H, η ,	O, o , omicron	Ψ , ψ , psi
Θ , θ , ϑ ,	Π , π , pi	Ω , ω , omega.

Θ

ANALYTIC GEOMETRY

CHAPTER I

CARTESIAN COORDINATES

1. THEOREM. *Let $X'OX$, $Y'OY$ be two fixed straight lines in a plane, then the position of any point in the plane is determined by its distance from $Y'OY$ measured parallel to $X'OX$, and its distance from $X'OX$ measured parallel to $Y'OY$.*

Let P be a point in the plane of $X'OX$, $Y'OY$, whose distance to the right of $Y'OY$, measured parallel to $X'OX$ is a , and whose distance above $X'OX$, measured parallel to $Y'OY$ is b . Then the position of P in the plane is determined. Because P must lie in the line MN , parallel to $Y'OY$ at the distance a (as defined above) from it on the right, and also in the line RS , parallel to $X'OX$ at the distance b from it on the upper side. Hence P must be at the intersection of MN and RS .

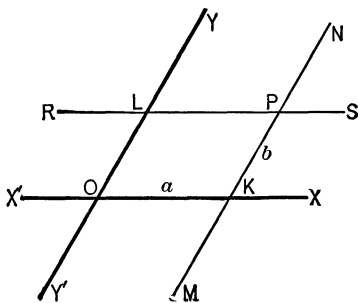


FIG. 1

DEFINITIONS. The two lengths or distances which determine the position of a point, as a and b determine P in the preceding demonstration, are called the **coordinates of the point**.

The fixed lines in the plane to which the position of the point is referred are called the **axes of coordinates**, and their intersection O is called the **origin of coordinates**, or simply the

origin. The axis $X'X$ is called the **axis of abscissas**, or the **X -axis**, and $Y'Y$ is called the **axis of ordinates**, or the **Y -axis**.

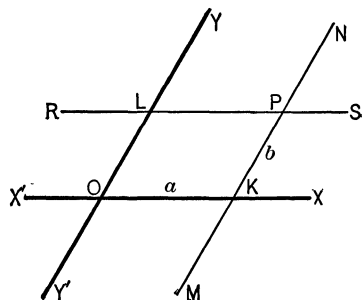


FIG. 1

The two coordinates a and b of the point P , Fig. 1, are called respectively the **abscissa** and the **ordinate** of the point. That is:

The abscissa of a point is its distance from the Y -axis, measured parallel to the X -axis.

The ordinate of a point is its distance from the X -axis, measured parallel to the Y -axis.

The letter x is customarily used to designate the abscissa of a point, and the letter y to designate the ordinate. So also the expression "the x of a point" is often used, meaning its abscissa, or "the y of a point," meaning its ordinate.

The coordinate axes are usually taken at right angles to each other, and in this book they will always be considered as so situated unless the contrary is stated. When the axes are not at right angles the coordinate system is said to be oblique.

NOTE. The term *Cartesian coordinates* is applied to the system of coordinates which has just been described because it is essentially the system invented by the French philosopher and mathematician Descartes (1596–1650). Other coordinate systems are often used, one of which, the polar system, will be described in a later chapter of this book.

2. Signs of the coordinates. Notation. In order to be able to locate a point when its coordinates are given we must know not only the lengths of its coordinates but also the direction (right or left, up or down) in which they are to be measured. Thus, Fig. 2, if $NO = OM = a$, and $RO = OS = b$, there are four points P_1, P_2, P_3, P_4 at the distance a from $Y'Y$ and the

distance b from $X'X$. To avoid this ambiguity we adopt the usual convention of distinguishing between lengths measured in opposite directions by $+$ and $-$ signs. Then, Fig. 3,

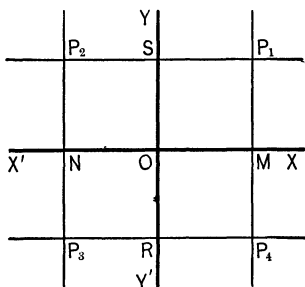


FIG. 2

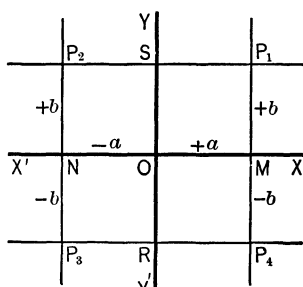


FIG. 3

if OM is called $+a$, ON is $-a$; and if OS is $+b$, OR is $-b$. Hence the coordinates of P_1 are $x = +a$, $y = +b$, those of P_2 are $x = -a$, $y = +b$, those of P_3 are $x = -a$, $y = -b$, and those of P_4 are $x = +a$, $y = -b$.

In actual practice when an abscissa or ordinate is $+$ the sign is usually not written.

In designating the position of a point by its coordinates the notation (x, y) is used. That is, the coordinates are written in parenthesis with a comma separating them, and the abscissa is invariably written first. Thus

the point $(2, -3)$ means the point whose abscissa is 2 and ordinate -3 , the point $(-3, 2)$ is the point whose abscissa is -3 and ordinate 2. See Fig. 4.

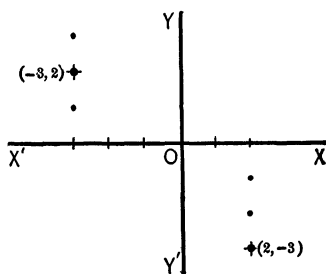


FIG. 4

EXERCISES. 1. Draw a pair of coordinate axes and, using any

convenient unit of length, locate the following points (1, 2), (5, -2), (3, 3), (-3, 1), (-2, 0), (0, 4), (-4, -2), (3, 0), (0, -2), (0, 0).

2. If a point is on the X -axis what is its ordinate?
3. If a point is on the Y -axis what is its abscissa?
4. What are the coordinates of the origin?
5. If the abscissa of a point is 0, on what line will it lie?
6. If the ordinate of a point is 0, on what line will it lie?

3. Directed lengths. By a directed length is meant a length measured in a given direction. As has been explained the coordinates of a point are always directed lengths. The length AB means the distance from A to B in magnitude and direction.

I. If A and B are two points, then

$$\text{length } AB = -\text{length } BA.$$

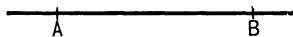


FIG. 5

That is, the magnitude element of the length is the same, whether we think of AB or of BA , but the direction element in the one case is opposite to that in the other.

II. THEOREM. *If three points A, B, C are arranged in any order on a straight line then $AC = AB + BC$.*

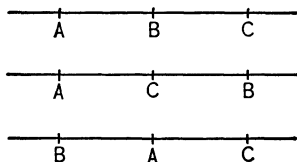


FIG. 6

In the upper line, Fig. 6, the three segments AC, AB, BC are all measured in the same direction, and AC is composed of the other two. Hence $AC = AB + BC$.

In the second line of the figure $AC = AB - CB$, but from I, $-CB = +BC$, hence $AC = AB + BC$.

In the third line $AC = BC - BA = AB + BC$, as before.

Other possible arrangements can be similarly treated.

III. PROBLEM. *To express the distance from one point to another, when the distances of these two points from a third point on the same line are known.*

Let the distances AB and AC , Fig. 6, be known in both length and direction, and let the distance from B to C be required. It is then only necessary to express the length of the required segment in terms of the given segments as in II, being careful to write the letters in the proper order in each case. Since it is here required to find the distance *from B to C* we write BC , and then, according to II,

$$BC = BA + AC,$$

and hence

$$BC = -AB + AC. \quad (1)$$

Had the question been to find the distance from C to B , the same procedure would lead to the equation

$$CB = CA + AB,$$

or

$$CB = -AC + AB. \quad (2)$$

The student should examine this discussion in connection with each arrangement of points in Fig. 6 and satisfy himself that the truth of the result is independent of the order of the points.

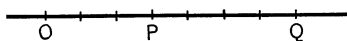


FIG. 7

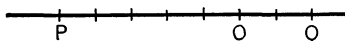


FIG. 8

ILLUSTRATIONS. In Fig. 7 let $OP = 3$, and $OQ = 7$, then

$$PQ = PO + OQ = -3 + 7 = 4.$$

Again in Fig. 8 suppose $OP = -5$, and $OQ = 2$, then

$$PQ = PO + OQ = -(-5) + 2 = 7.$$

Similarly with the same data

$$QP = QO + OP = -2 + (-5) = -7.$$

④ Distance between two points.

PROBLEM. *To find the length of a line joining two points whose coordinates are given, the axes being rectangular.*

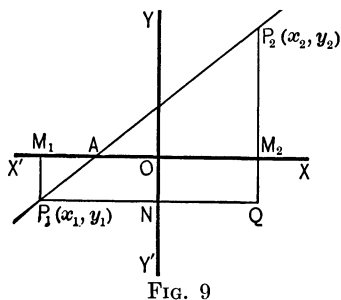


FIG. 9

Let it be required to express the length of P_1P_2 , in terms of the given coordinates (x_1, y_1) , (x_2, y_2) of these two points. That is we have given $OM_1 = x_1$, $M_1P_1 = y_1$, $OM_2 = x_2$, $M_2P_2 = y_2$.

Draw P_1Q parallel to OX to meet M_2P_2 (extended in this case) at Q . Since P_1QP_2

is a right angle,

$$\overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2. \quad (i)$$

But

$$P_1Q = P_1N + NQ = -OM_1 + OM_2 = -x_1 + x_2,$$

and

$$QP_2 = QM_2 + M_2P_2 = -M_1P_1 + M_2P_2 = -y_1 + y_2.$$

Substituting these values in (i) the result is

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2,$$

or since $(x_2 - x_1)^2 = (x_1 - x_2)^2$, and $(y_2 - y_1)^2 = (y_1 - y_2)^2$,

$$P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \quad (3)$$

Note carefully that in indicating a line-segment, simply as a length, it is to be written in the positive direction, as P_1Q ; QP_2 in the foregoing demonstration; but in indicating a line-segment as one of the coordinates of a point it must be taken as starting from the axis and going to the point. Thus, above, OM_1 (not M_1O) is x_1 , and M_1P_1 (not P_1M_1) is y_1 .

⑤ Slope of a line.

I. DEFINITION. *The tangent of the angle which a line makes with the X-axis is called the **slope** of the line.*

Of the four angles formed by the intersection of a line with the X -axis that one is taken as the "slope" angle which lies between the positive extension of the X -axis and the upper extension of the line. Thus the slope of AB is $\tan \theta_1$ and the slope of CD is $\tan \theta_2$.

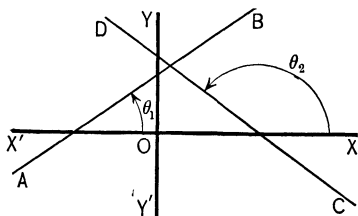


FIG. 10

II. PROBLEM. *To find the slope of a line in terms of the coordinates of two points on the line.*

The slope of the line determined by the points P_1, P_2 Fig. 9 is $\tan XAP_2 = \tan QP_1P_2$. Now

$$\tan QP_1P_2 = \frac{QP_2}{P_1Q},$$

and as shown in Art. 4, $QP_2 = -y_1 + y_2$, and $P_1Q = -x_1 + x_2$,

$$\therefore \tan QP_1P_2 = m = \frac{-y_1 + y_2}{-x_1 + x_2} = \frac{y_2 - y_1}{x_2 - x_1}, \quad (4)$$

where m is used to designate the slope.

Since in deriving (3) and (4) the principles of directed lengths have been strictly followed the results are true for all positions of the points P_1 and P_2 .

⑥ Division of a segment of a line in a given ratio.

PROBLEM. *To find the coordinates of the point which divides a given segment of a line in a given ratio.*

Some preliminary remarks are necessary.

(a) By a given segment of a line is meant a segment the coordinates of whose extremities are known.

(b) A segment may be divided internally or externally. Thus in Fig. 11 (a) the point P divides the segment AB in-